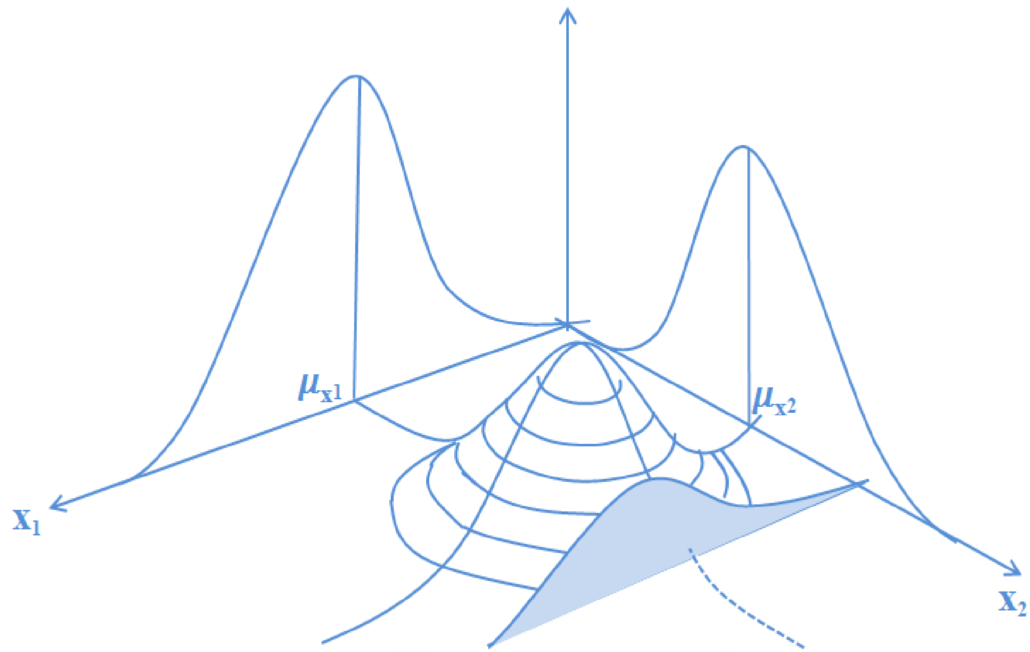
New methods for time-variant reliability assessment of degrading structures



Lara HAWCHAR

Franck SCHOEFS

Charbel-Pierre EL SOUEIDY

Outline

- ✓ Problem context
- ✓ Time-variant reliability analysis
- ✓ Polynomial chaos expansion
- ✓ Application examples
- ✓ Conclusion and work in progress

Problem context

Design ⟶ **Monitoring** ⟶ **Maintenance**

Uncertainty

Material properties

Geometry parameters

Loadings

Time dependency

Degradation phenomena (*e.g. corrosion, fatigue, cracking, ...*)

Time-variant loadings (*e.g. wind, swell, traffic, ...*)

Problem context

Design → **Monitoring** → **Maintenance**

Uncertainty

Material properties

Geometry parameters

Loadings

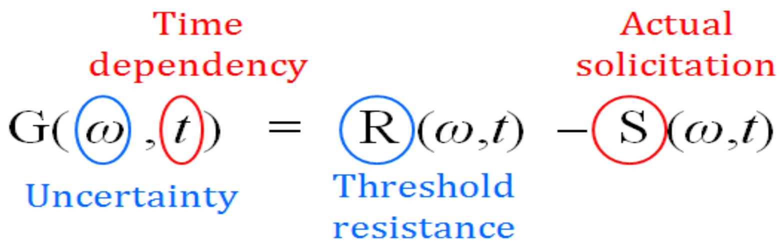
Time-variant reliability analysis

Time dependency

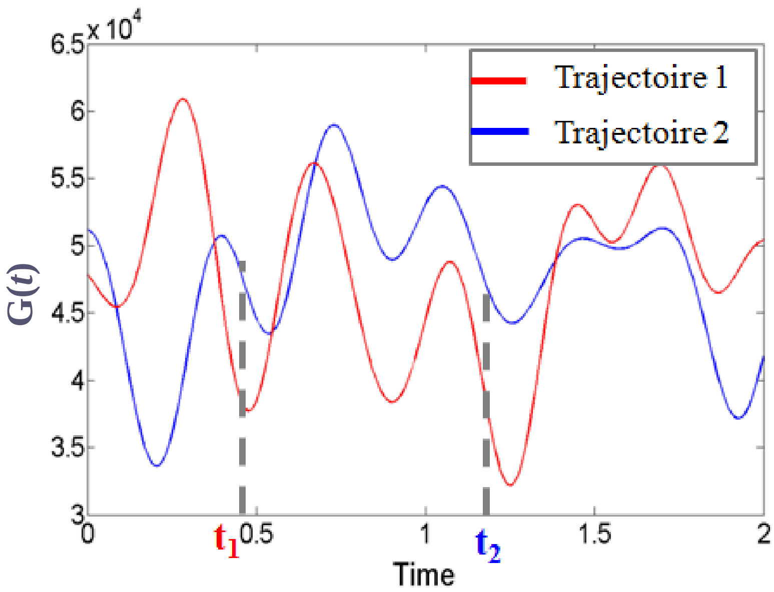
Degradation phenomena (*e.g. corrosion, fatigue, cracking, …*)

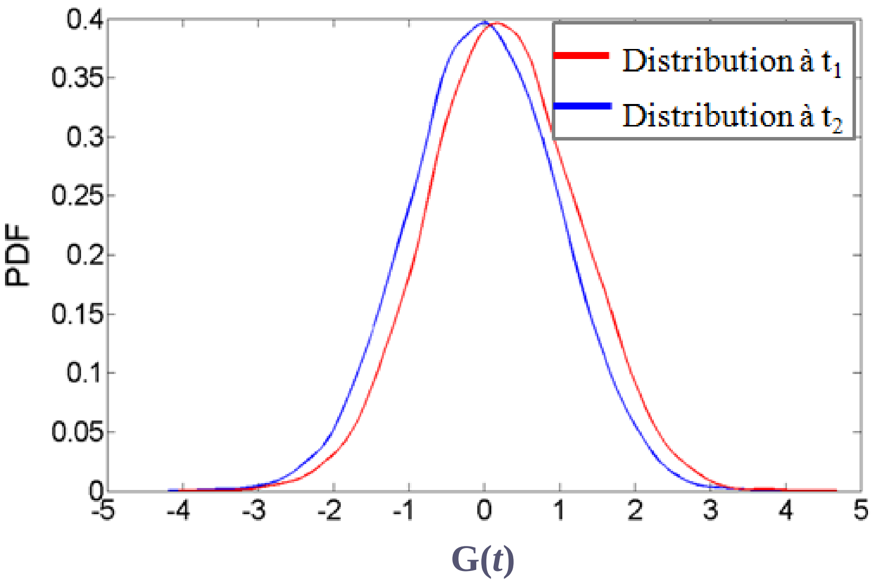
Time-variant loadings (*e.g. wind, swell, traffic, …*)

Time-dependent Limit State Function



$$G(\omega, t) = R(\omega, t) - S(\omega, t)$$

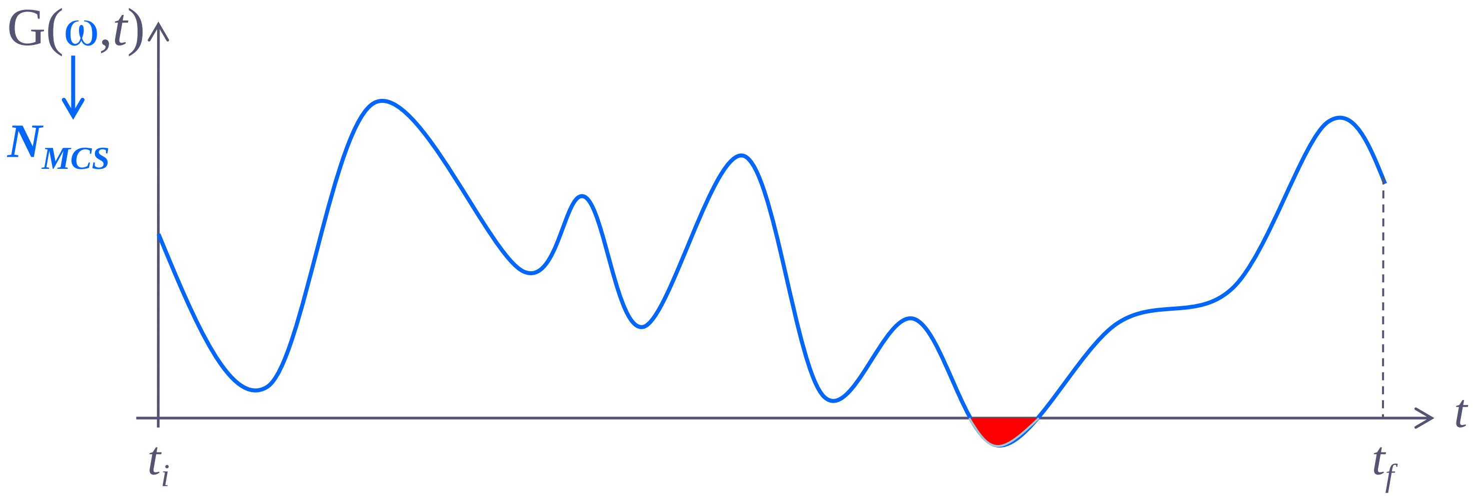




Time-variant reliability problem

$$P_{f,c}(t_i,t_f) = \text{Prob}\,(\exists\ \tau \in [t_i,t_f] : G(\omega\,,\tau) < 0)$$

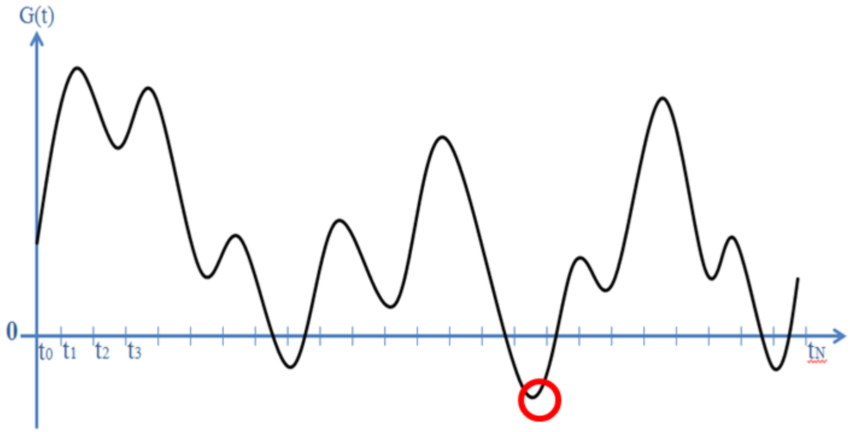
Monte-Carlo Simulation (MCS) method

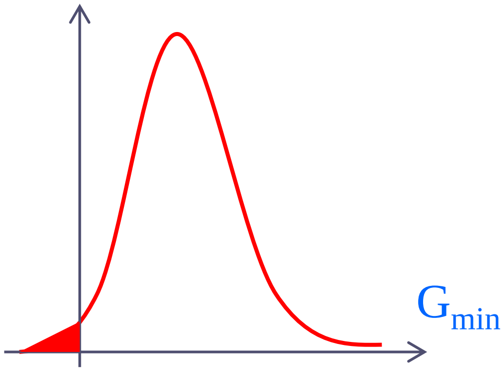


Prohibitive !!

$$P_{f,c}\left(t_i,t_f\right) \approx \frac{\text{Number of failing trajectories}}{\text{Total number of trajectories}}$$

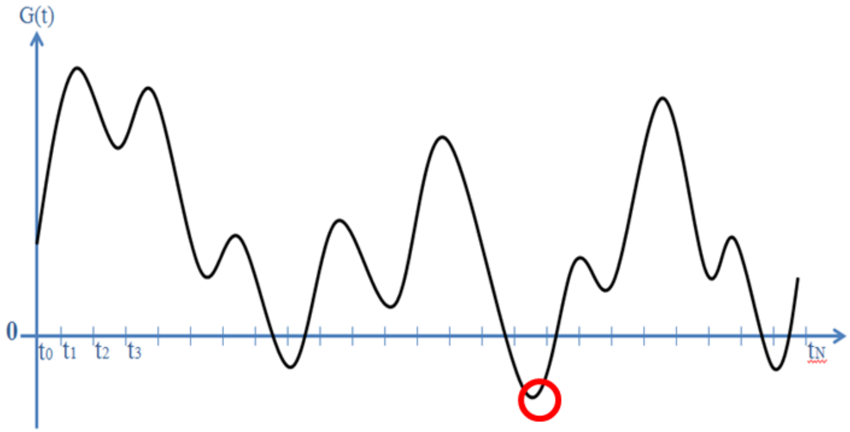
Extreme performance approach

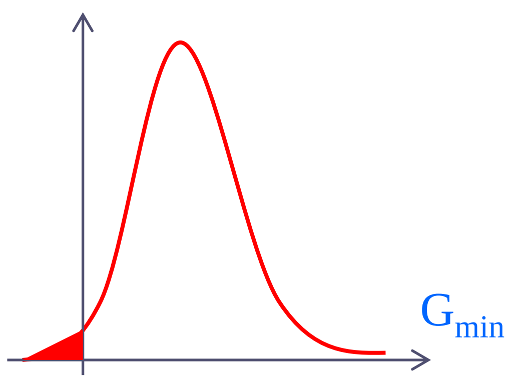




$$P_{f,c}(t_i,t_f) \approx \text{Prob}\,(\min_{t_i \le \tau \le t_f}\{G(\tau)\} < 0)$$

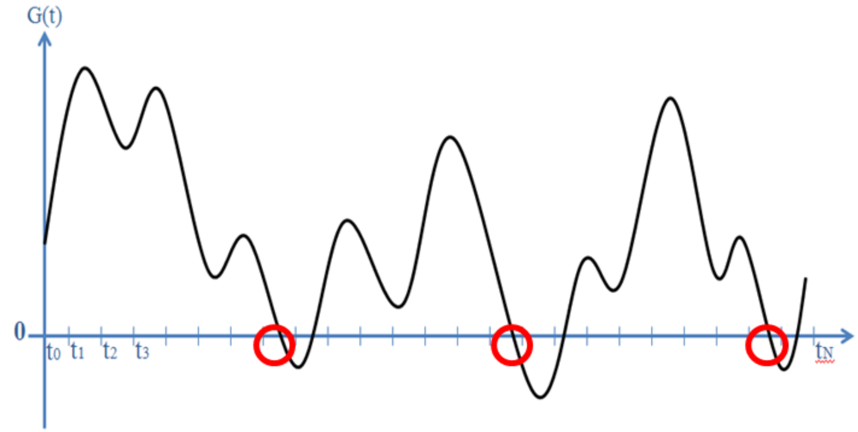
Extreme performance approach





$$P_{f,c}(t_i,t_f) \approx \text{Prob}\,(\min_{t_i \leq \tau \leq t_f} \{G(\tau)\} < 0)$$

Outcrossing approach



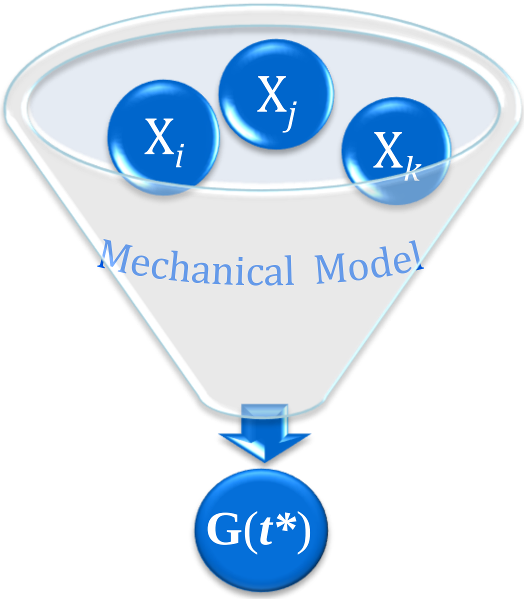
$$\upsilon^+(t) = \lim_{\Delta t \to 0^+} \frac{\text{Prob}(\{G(t) > 0\} \cap \{G(t+\Delta t) < 0\})}{\Delta t}$$

$$P_{f,c}(t_i,t_f) \leq P_{f,i}(t_0) + E[N^+(t_i,t_f)]$$

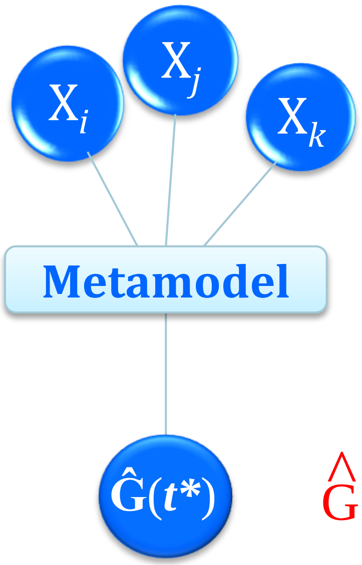
"PHI2 method"

Andrieu-Renaud C., Sudret, B. and Lemaire, L. (2004). "The PHI2 method: a way to compute time-variant reliability." Reliability Engineering and system Safety. 84: 75-86.

Metamodeling techniques

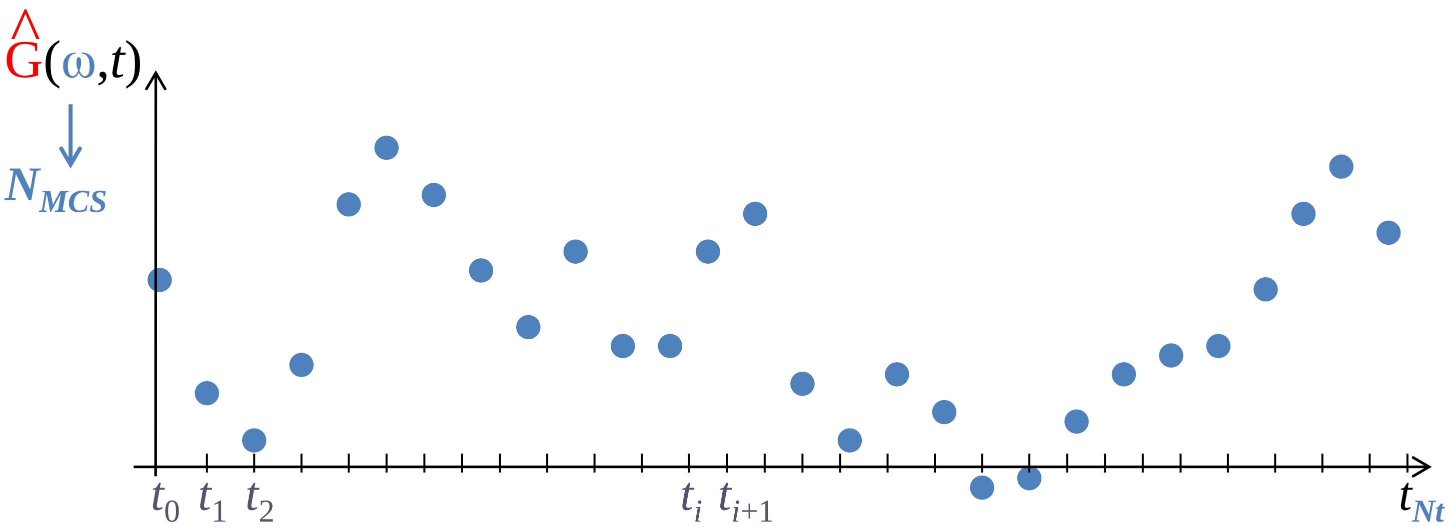


Metamodeling techniques



$N_s << N_{MCS}$

Number of evaluations of G(t)= $N_s \times N_t$



Polynomial Chaos Expansions

unknown coefficients … polynomial chaos basis

$$G(\mathbf{X}) \approx \hat{G}^{(CP)}(\xi) = a_0\,\psi_0(\xi) + a_1\psi_1(\xi) + a_2\psi_2(\xi) + a_3\psi_3(\xi) + \ldots + a_{(P\text{-}1)}\,\psi_{(P\text{-}1)}(\xi)$$

$$\begin{pmatrix} X_1^{(1)} & X_2^{(1)} & \cdots & X_m^{(1)} \\ \vdots & & \ddots & \vdots \\ X_1^{(N_s)} & X_2^{(N_s)} & \cdots & X_m^{(N_s)} \end{pmatrix}$$

Original model →

$$\begin{pmatrix} Y^{(1)} \\ \vdots \\ Y^{(N_s)} \end{pmatrix}$$

Isoprobabilistic transformation ↓

$$\begin{pmatrix} \xi_1^{(1)} & \xi_2^{(1)} & \cdots & \xi_m^{(1)} \\ \vdots & & \ddots & \vdots \\ \xi_1^{(N_s)} & \xi_2^{(N_s)} & \cdots & \xi_m^{(N_s)} \end{pmatrix}$$

Surrogate model →

$$\begin{pmatrix} \sum_{a \in A} c_a \Psi_a\left(\xi_1^{(1)}, \xi_2^{(1)}, \ldots, \xi_m^{(1)}\right) \\ \vdots \\ \sum_{a \in A} c_a \Psi_a\left(\xi_1^{(N_s)}, \xi_2^{(N_s)}, \ldots, \xi_m^{(N_s)}\right) \end{pmatrix}$$

Regression

Polynomial Chaos Expansions

unknown coefficients — polynomial chaos basis

$$G(\mathbf{X}) \approx \hat{G}^{(CP)}(\xi) = \cancel{a_0 \psi_0(\xi)} + a_1 \psi_1(\xi) + \cancel{a_2 \psi_2(\xi)} + a_3 \psi_3(\xi) + \ldots + \cancel{a_{(P\text{-}1)} \psi_{(P\text{-}1)}(\xi)}$$

$$\begin{pmatrix} X_1^{(1)} & X_2^{(1)} & \cdots & X_m^{(1)} \\ \vdots & & \ddots & \vdots \\ X_1^{(N_s)} & X_2^{(N_s)} & \cdots & X_m^{(N_s)} \end{pmatrix} \xrightarrow{\text{Original model}} \begin{pmatrix} Y^{(1)} \\ \vdots \\ Y^{(N_s)} \end{pmatrix}$$

Isoprobabilistic transformation ↓

$$\begin{pmatrix} \xi_1^{(1)} & \xi_2^{(1)} & \cdots & \xi_m^{(1)} \\ \vdots & & \ddots & \vdots \\ \xi_1^{(N_s)} & \xi_2^{(N_s)} & \cdots & \xi_m^{(N_s)} \end{pmatrix} \xrightarrow{\text{Surrogate model}} \begin{pmatrix} \sum_{a \in A} c_a \Psi_a\left(\xi_1^{(1)}, \xi_2^{(1)}, \ldots, \xi_m^{(1)}\right) \\ \vdots \\ \sum_{a \in A} c_a \Psi_a\left(\xi_1^{(N_s)}, \xi_2^{(N_s)}, \ldots, \xi_m^{(N_s)}\right) \end{pmatrix}$$

Regression

Hyperbolic truncation

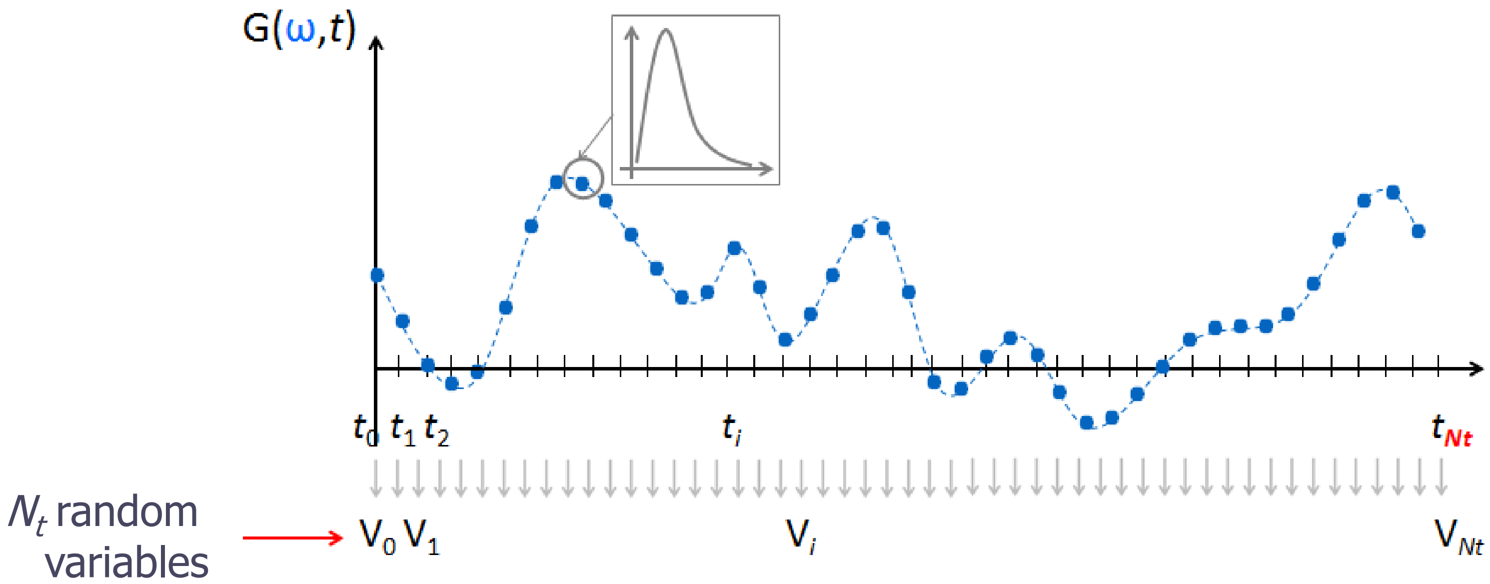
$$A_\beta = \left\{ \boldsymbol{\alpha} \in \mathbf{N}^{m:} : \|\boldsymbol{\alpha}\|_\beta = \left(\sum_{i=1}^{m} \alpha_i^\beta \right)^{1/\beta} \leq p \right\}$$

Adaptive regression-based algorithm

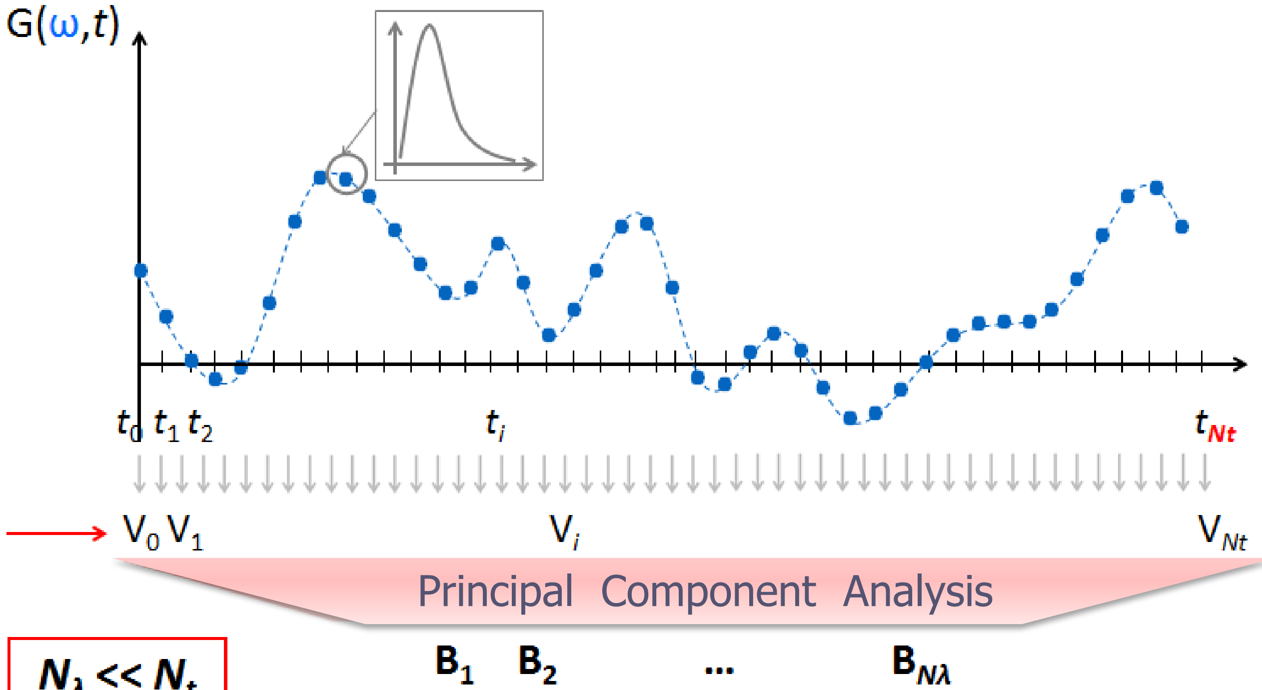
R^2: Local error

Q^2: Global error

Time-variant limit state function



Time-variant limit state function



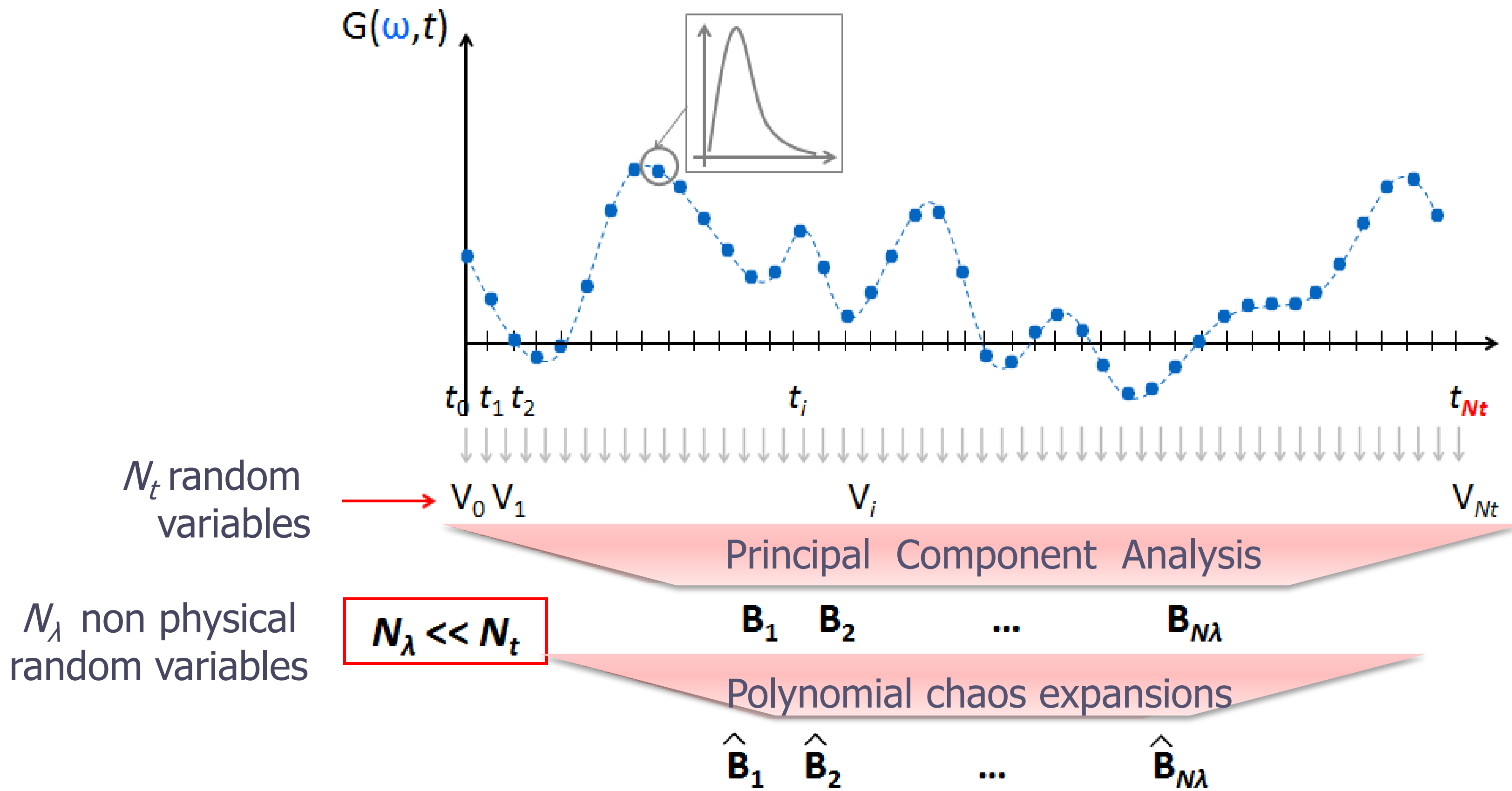
N_t random variables

N_λ non physical random variables

$N_\lambda << N_t$

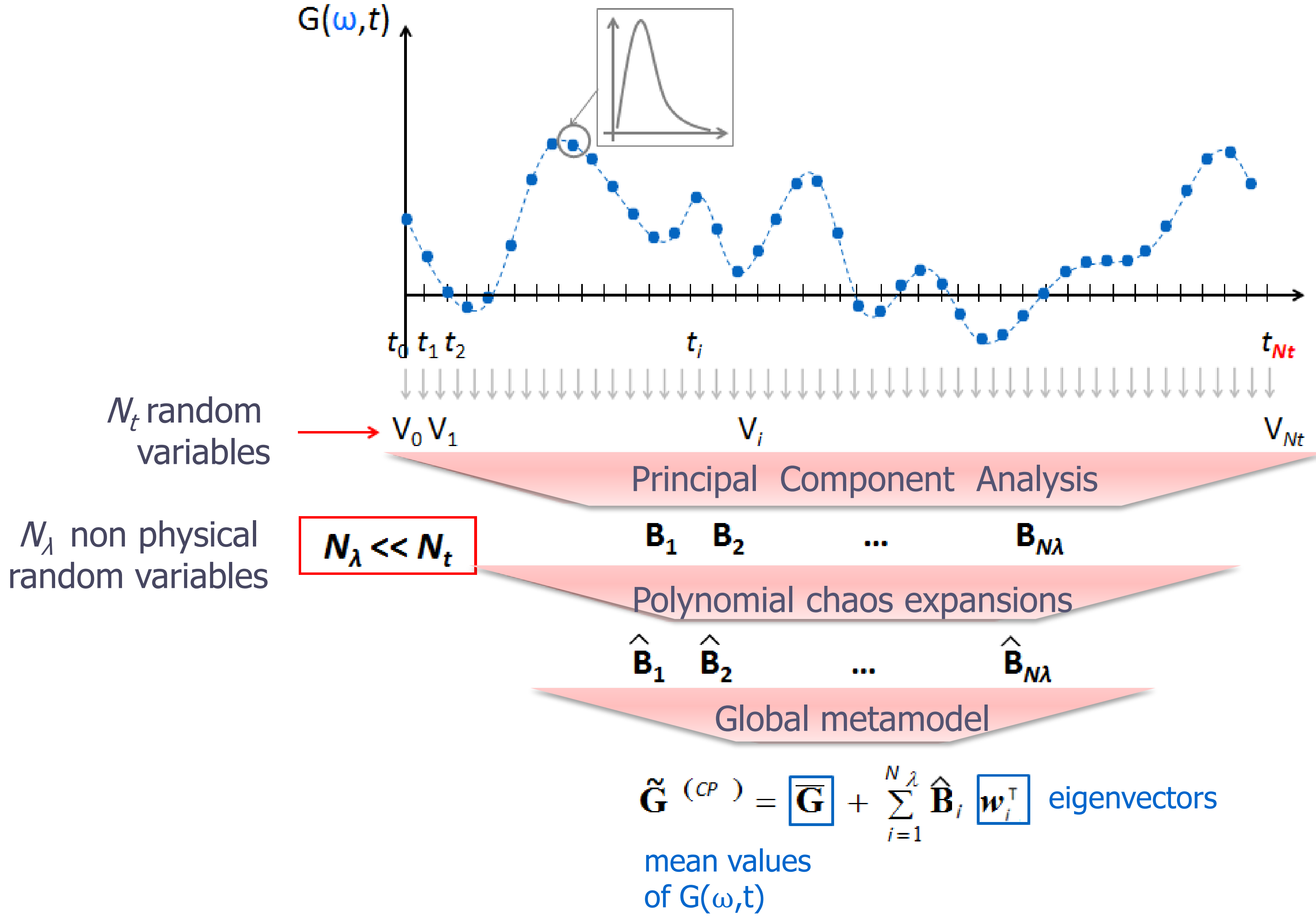
Polynomial Chaos Expansions for Time-Variant Reliability Analysis

Time-variant limit state function



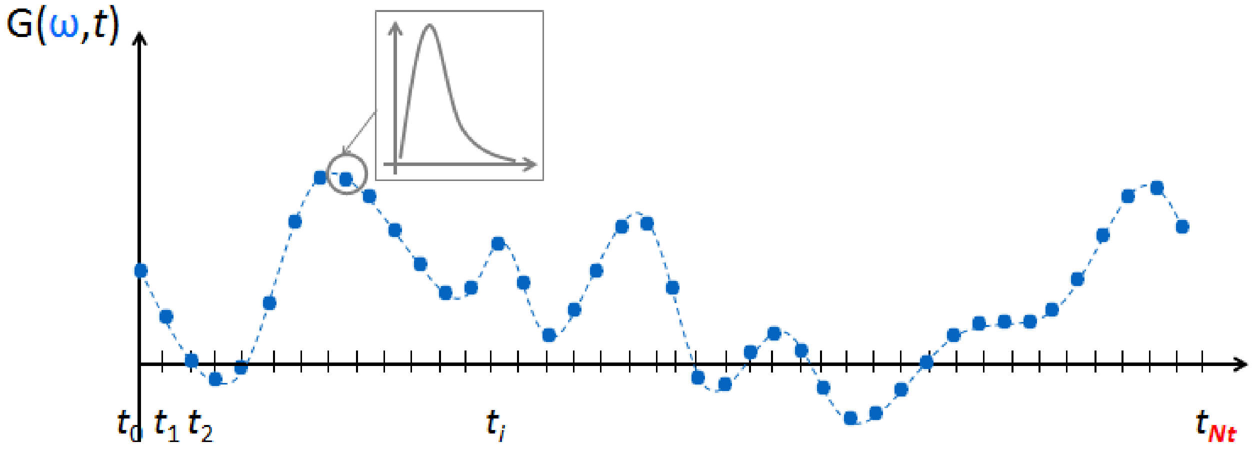
Polynomial Chaos Expansions for Time-Variant Reliability Analysis

Time-variant limit state function



$$\tilde{G}^{(CP)} = \overline{G} + \sum_{i=1}^{N_\lambda} \hat{B}_i \, w_i^T$$

Time-variant limit state function



Inputs

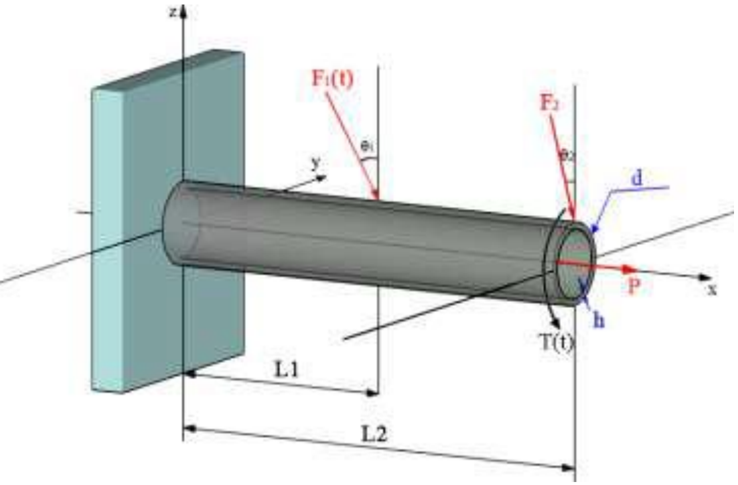
Outputs

Standard Gaussian random variables $\xi_1, \xi_2, \ldots$ → $\hat{G} = \overline{G} + \sum_{i=1}^{N_\lambda} \hat{B}_i \; w_i^t$ → $G(t_0), G(t_1), \ldots, G(t_f)$

$$P_{f,c}(t_i,t_f) \approx \frac{\text{Number of failing trajectories}}{\text{Total number of trajectories}}$$

Illustrative Examples

Case study 1



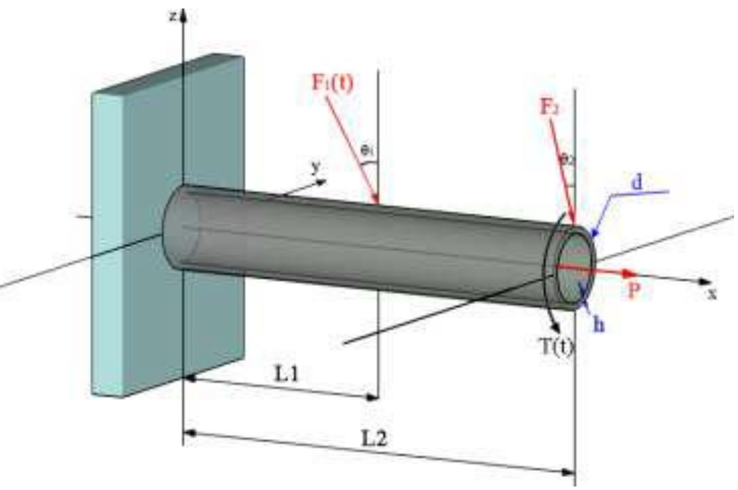
Parameter	Distribution	Mean	Coefficient of variation	Autocorrelation function
L_1	Deterministic	60 mm	0 %	NA
L_2	Deterministic	120 mm	0 %	NA
θ_1	Deterministic	10 deg	0 %	NA
θ_2	Deterministic	5 deg	0 %	NA
d	Normal	42 mm	1.19 %	NA
h	Normal	5 mm	2 %	NA
R_0	Normal	560 MPa	10 %	NA
$F_1(t)$	**Gumbel Process**	**1800 exp(0.3*t*) N**	**10 %**	**$\exp[-\lvert\Delta t\rvert/4]$**
F_2	Normal	1800 N	10 %	NA
F_3	Gumbel	1000 N	10 %	NA
$T(t)$	**Gaussian Process**	**1900 N.m**	**10 %**	**$\exp[-(\Delta t/0.5)^2]$**

$$G(t) = R(t) - \sigma_{max}(t)$$

$$R(t) = R_0(1-1.01t)$$

Illustrative Examples

Case study 1

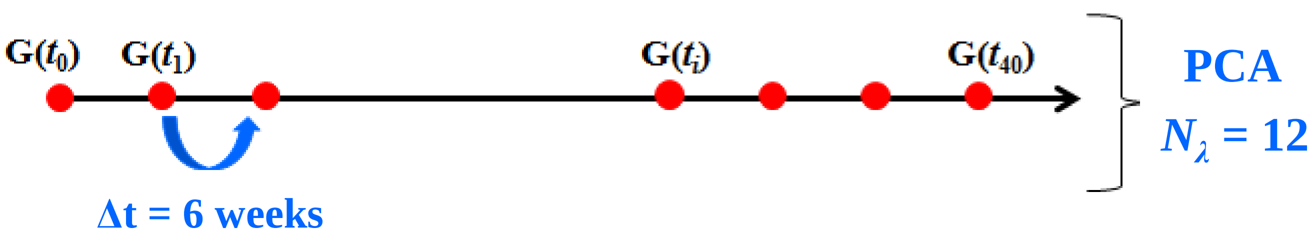


Parameter	Distribution	Mean	Coefficient of variation	Autocorrelation function
L_1	Deterministic	60 mm	0 %	NA
L_2	Deterministic	120 mm	0 %	NA
θ_1	Deterministic	10 deg	0 %	NA
θ_2	Deterministic	5 deg	0 %	NA
d	Normal	42 mm	1.19 %	NA
h	Normal	5 mm	2 %	NA
R_0	Normal	560 MPa	10 %	NA
$F_1(t)$	**Gumbel Process**	**1800 exp(0.3t) N**	**10 %**	**$\exp[-\lvert\Delta t\rvert/4]$**
F_2	Normal	1800 N	10 %	NA
F_3	Gumbel	1000 N	10 %	NA
$T(t)$	**Gaussian Process**	**1900 N.m**	**10 %**	**$\exp[-(\Delta t/0.5)^2]$**

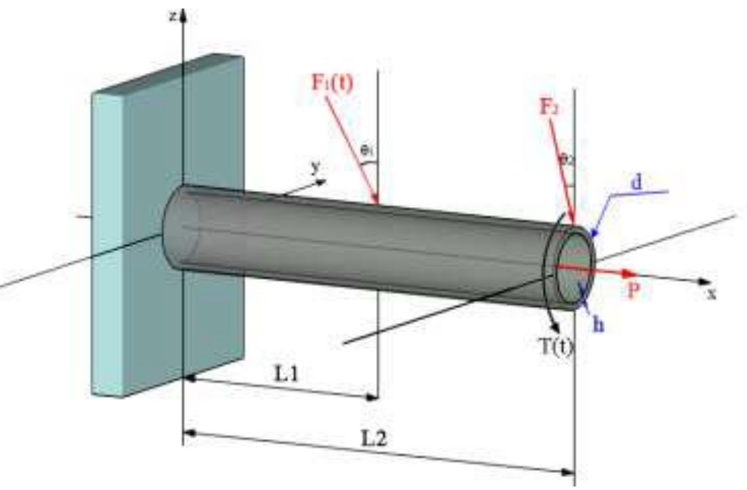
$$G(t) = R(t) - \sigma_{max}(t)$$

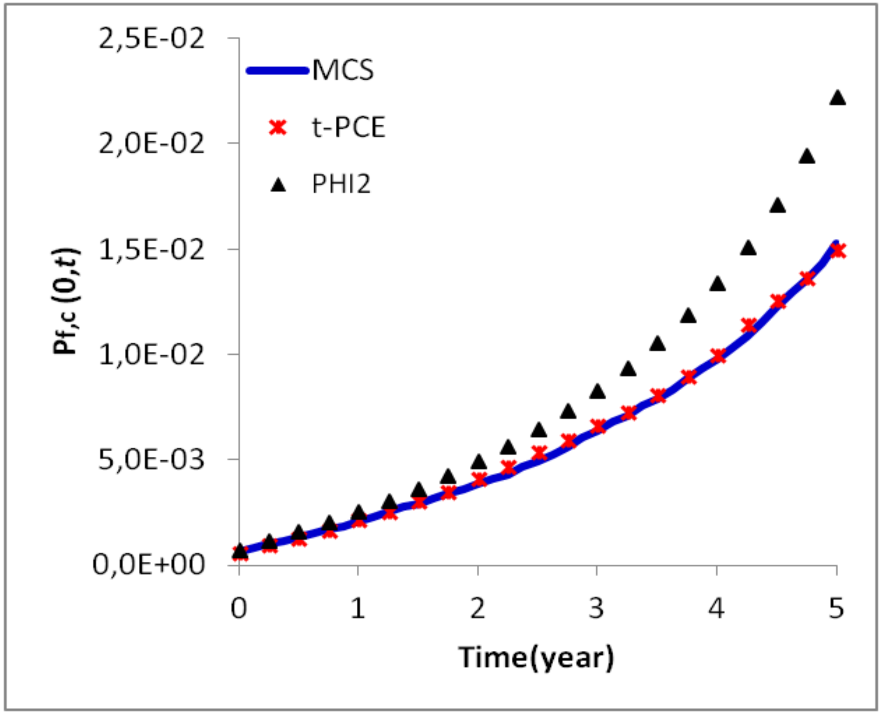
$$R(t) = R_0(1-0.01t)$$

Discretization of the stochastic processes → **28** input random variables in total



Case study 1





- Evolution in time of the cumulative probability of failure -

The proposed method is efficient for problems with Non-Gaussian Non-Stationary stochastic processes.

	MCS	*t*-PCE	PHI2
$P_{f,c}(0,5)$	[1.50 x 10^{-2} ; 1.55 x 10^{-2}]	1.50 x 10^{-2}	2.23 x 10^{-2}
$\varepsilon_{\%}$	---	**1.71 %**	**46.22 %**
number of function evaluations	41,000,000	**14,760**	**22,468**

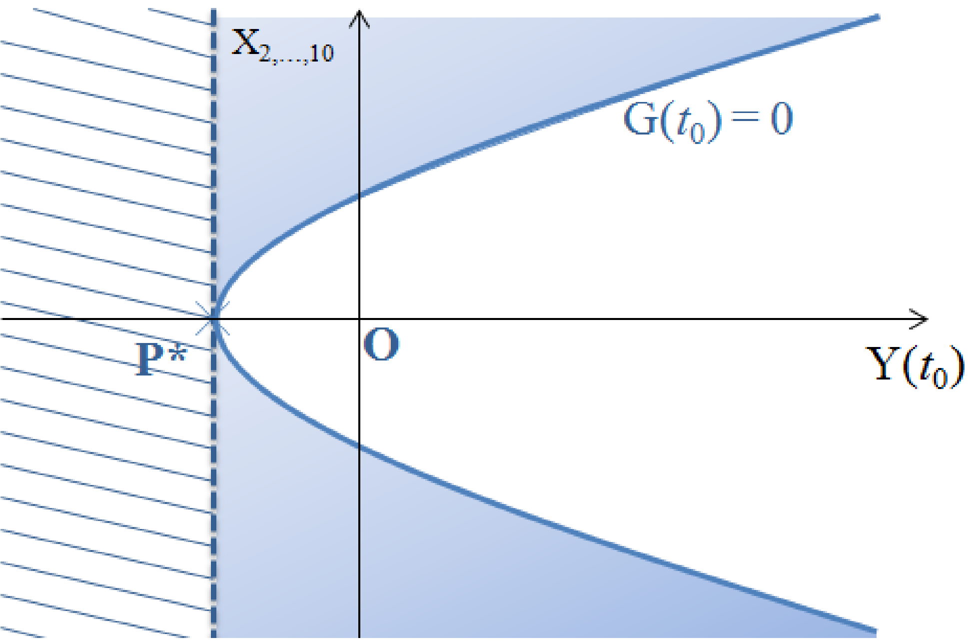
Case study 2

Highly non linear limit state function

$$G(t) = 3 + Y(t) - \frac{1}{6}\sum_{i=2}^{10} X_i^2$$

Gaussian Random process (Y(t))

Standard Gaussian Random variables (X_i)



- Shape of the limit state function at the initial time -

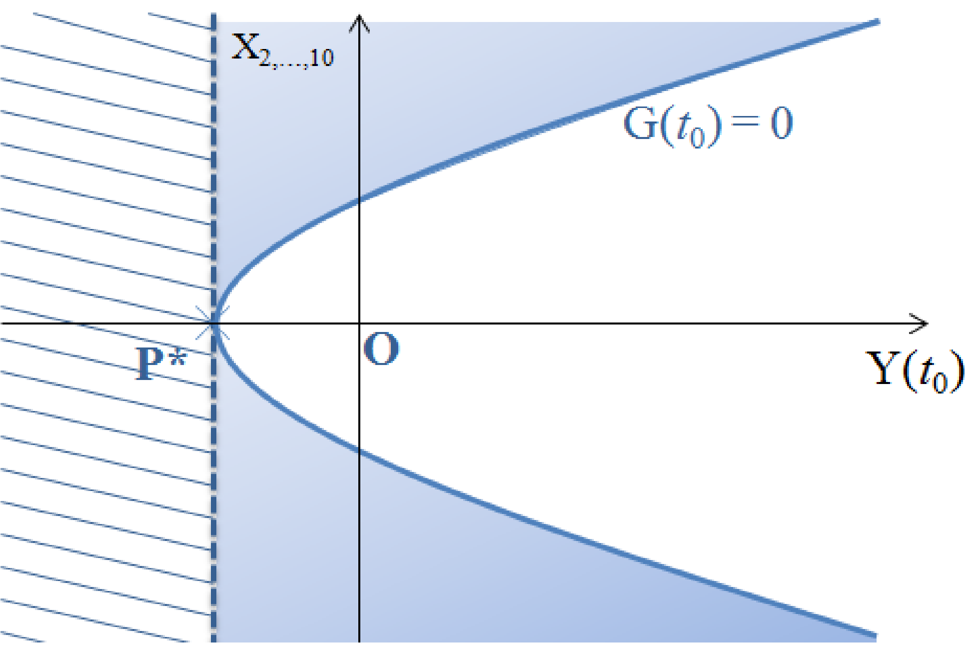
Case study 2

Highly non linear limit state function

$$G(t) = 3 + Y(t) - \frac{1}{6}\sum_{i=2}^{10} X_i^2$$

Gaussian Random process (→ $Y(t)$)

Standard Gaussian Random variables (→ X_i)



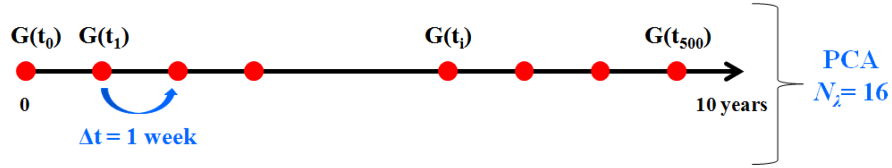
- Shape of the limit state function at the initial time -

$$\rho_F(t_1, t_2) = exp\left(-\left(\frac{t_2 - t_1}{1\,month}\right)^2\right)$$

$Y(t), X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}$

$\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6,$
$\xi_7, \xi_8, \xi_9, \xi_{10}, \xi_{11},$
$\xi_{12}, \xi_{13}, \xi_{14}, \xi_{15}$

24 random variables



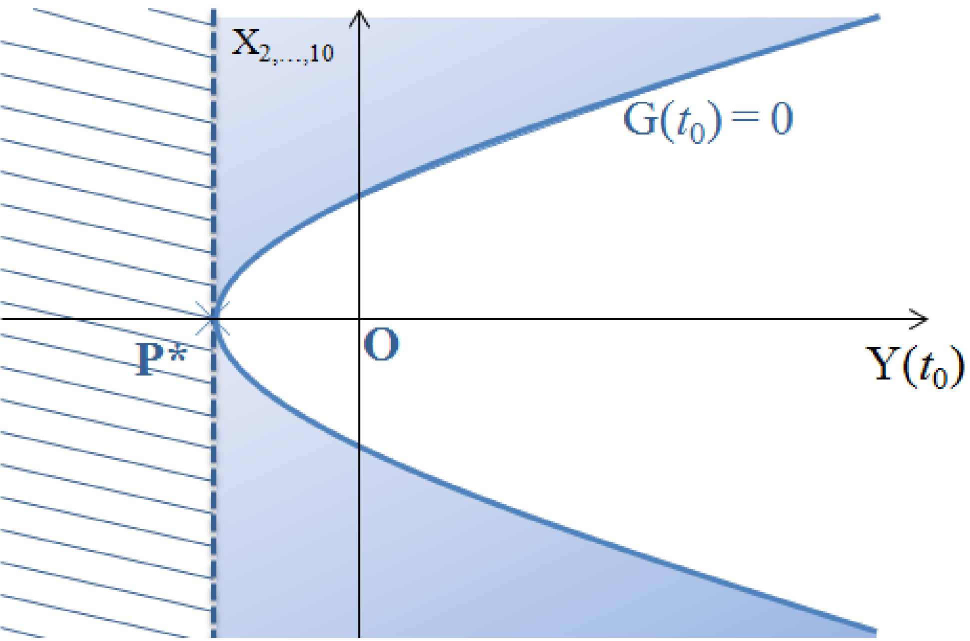
Case study 2

Highly non linear limit state function

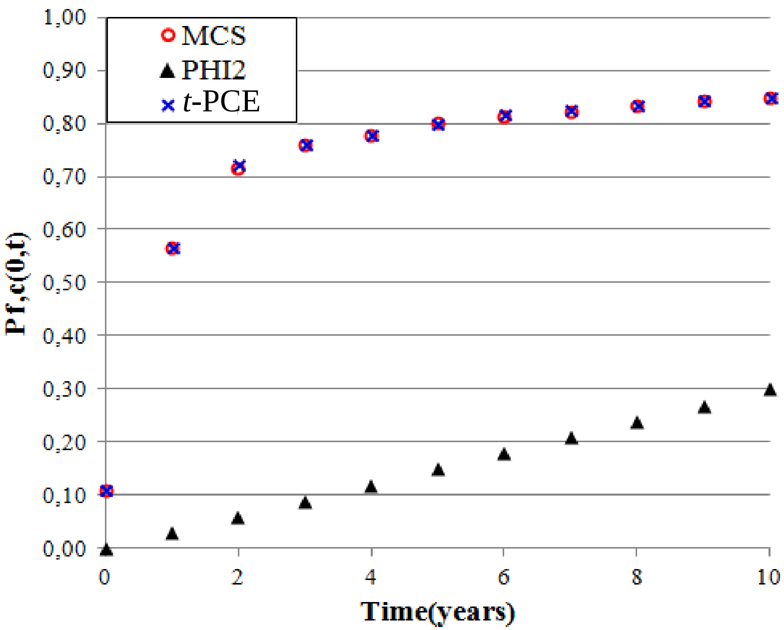
$$G(t) = 3 + Y(t) - \frac{1}{6}\sum_{i=2}^{10} X_i^2$$

Gaussian Random process

Standard Gaussian Random variables



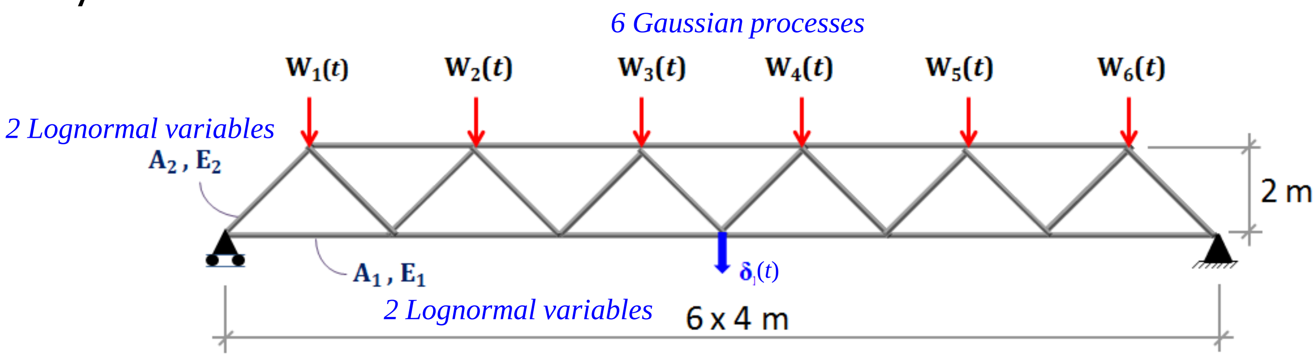
- Shape of the limit state function at the initial time -



- Evolution in time of the cumulative probability of failure -

Illustrative Examples

Case study 3

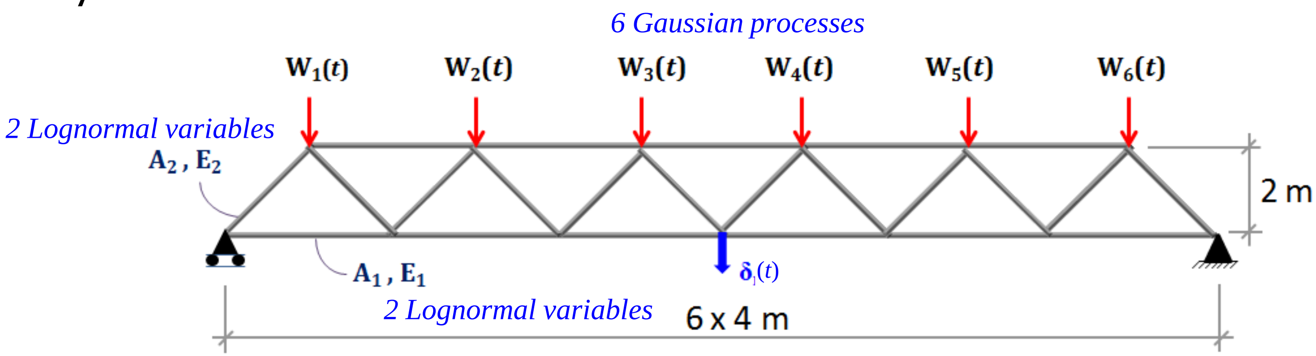


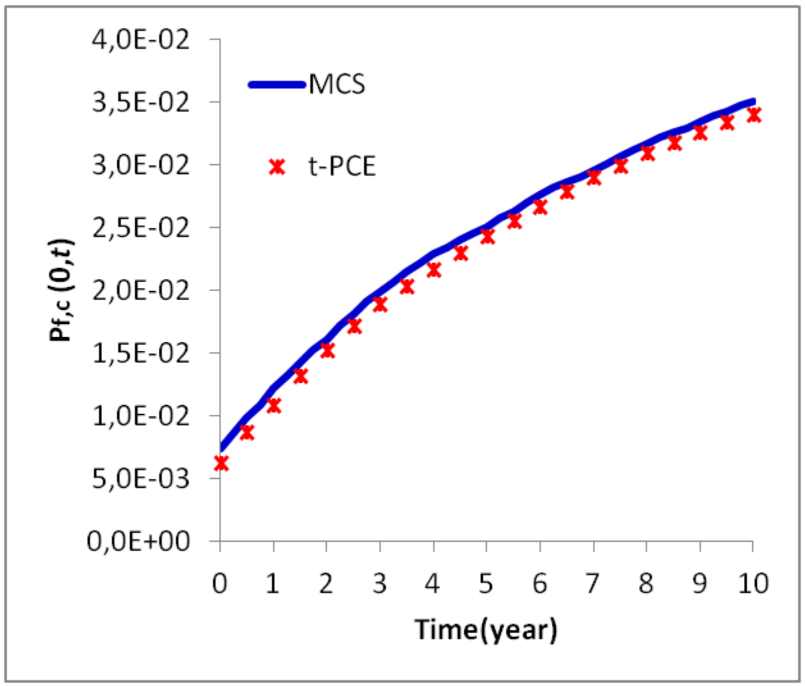
$G(t) = \delta_{max} - \delta(t)$ *"Finite Element Model"*

Discretization of the stochastic processes → **76** input random variables in total → **High dimensional problem**

Illustrative Examples

Case study 3





- Evolution in time of the cumulative probability of failure -

	MCS	*t*-PCE
$P_{f,c}(0, 5)$	[3.48 x 10^{-2} ; 3.55 x 10^{-2}]	3.40 x 10^{-2}
$\varepsilon_{\%}$	---	3.13 %
number of function evaluations	41,000,000	**32,800**

The proposed method is efficient for high dimensional problems.

Conclusion and Work in Progress

PCE *combined with PCA* is:

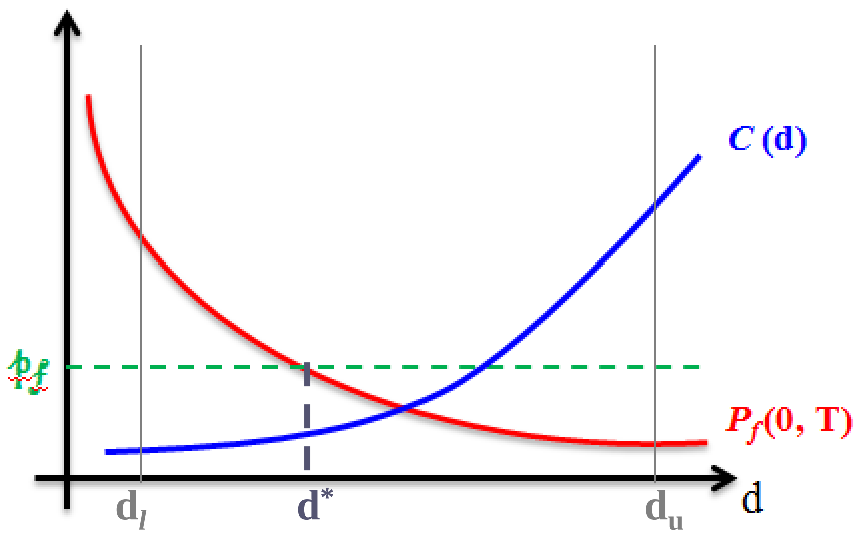
- ✓ efficient for time-variant reliability problems involving Non-Gaussian Non-Stationary stochastic processes.
- ✓better and more general than some recent methods (PHI2)
- ✓affordable for high dimensional problems
- ✓ suitable for highly nonlinear limit state functions

Conclusion and Work in Progress

Time-Variant Reliability-Based Design Optimization
(*t*-RBDO)

Time-variant reliability analysis aims to calculate the cumulative probability of failure of a given structure over its intended lifetime.

Time-variant reliability -based design optimization aims to find the optimal design (cost) of a structure while procuring a certain reliability level over the structure lifetime.



Time-Variant Reliability-Based Design Optimization
(*t*-RBDO)

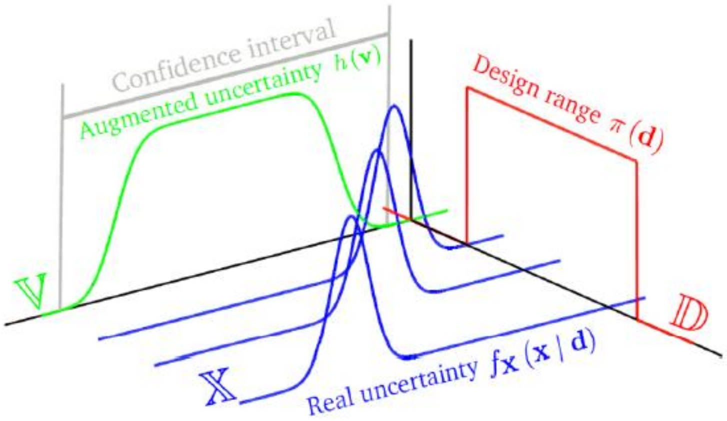
Minimize: $C(\mathbf{d})$ $\quad d_l \le d \le d_u$

Subject to: $P_f(0, t_i) \le \mathfrak{p}_f(t_i)$ $\quad 0 \le t_i \le \mathrm{T}$ with $i = 1, \ldots, N_t$ ➔ N_t probabilistic constraints

$\mathfrak{p}_f(t_i)$ is the threshold for P_f at t_i

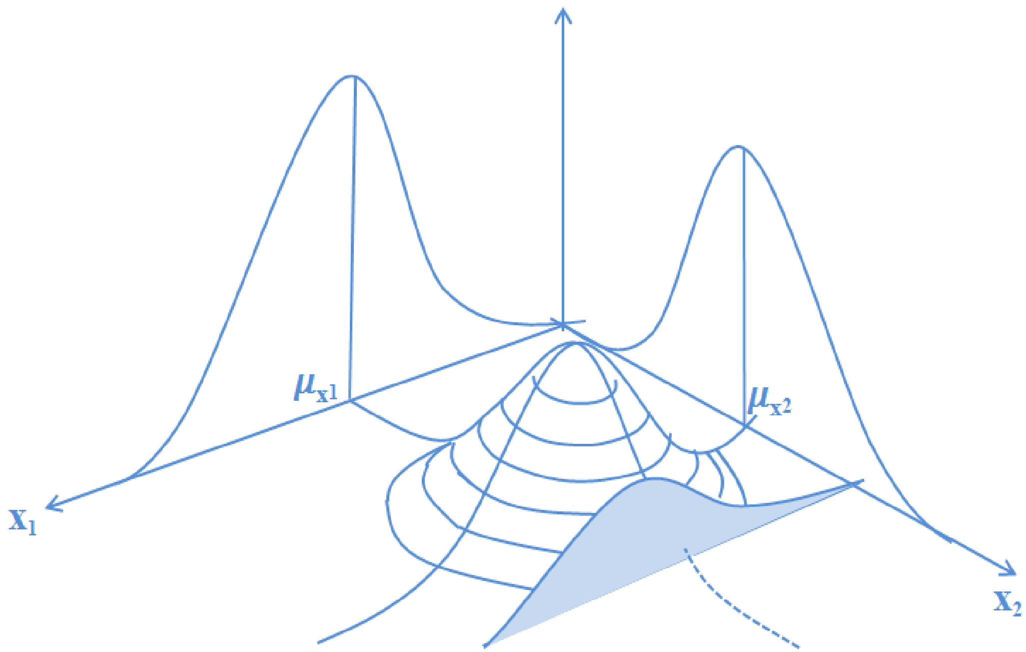
Augmented reliability space

$$h(\mathbf{x}) = \int_{\mathbb{D}} f_{\mathbf{X}}(\mathbf{x} \mid \mathbf{d})\, \pi(\mathbf{d})\, \mathrm{d}\mathbf{d}$$



New methods for time-variant reliability assessment of degrading structures

Thanks For Your Attention



Lara HAWCHAR

Franck SCHOEFS

Charbel-Pierre EL SOUEIDY

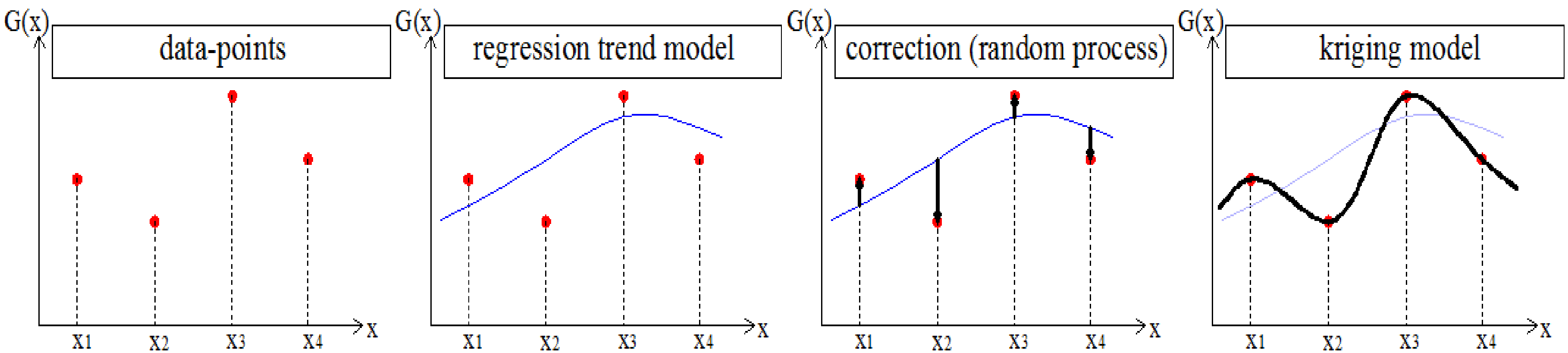
Conclusion and Work in Progress

KRIGING
(Gaussian Process Modeling)

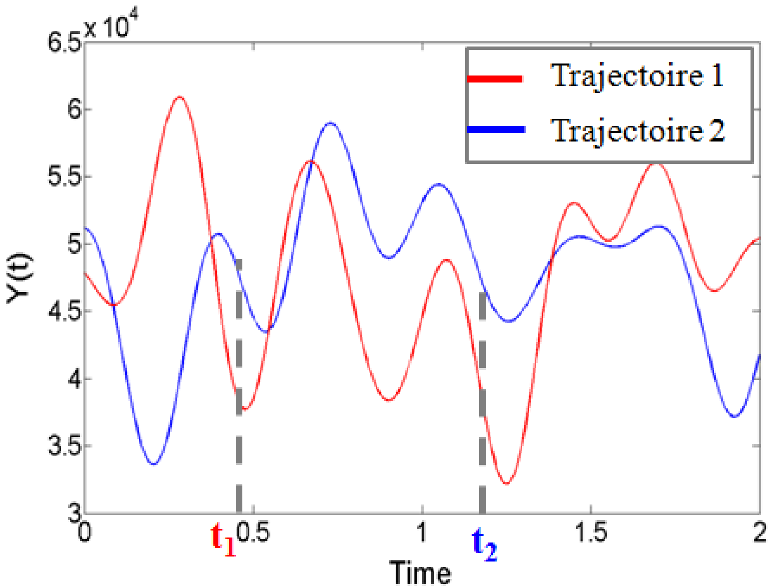
Regression model — Gaussian random process

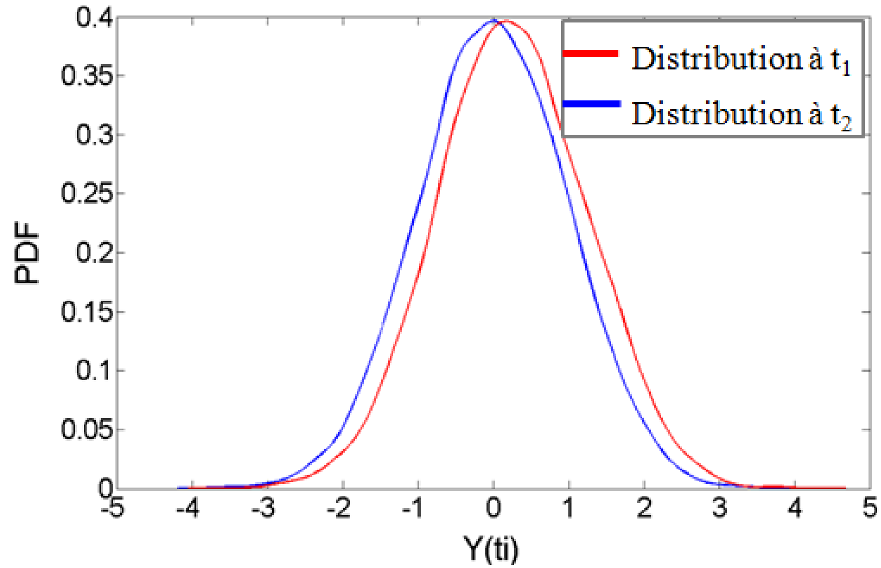
$$B_i(\mathbf{X}) \approx \hat{B}_i^{(K)}(\mathbf{X}) = f(\mathbf{X})^T \boldsymbol{\beta} + \sigma^2 Z(\mathbf{X})$$

- $f(\mathbf{X})$: Set of regression functions
- $\boldsymbol{\beta}$: Vector of unknown coefficients
- σ^2: Variance
- $Z(\mathbf{X})$: Standard Gaussian random process



Stochastic process $Y(t)$

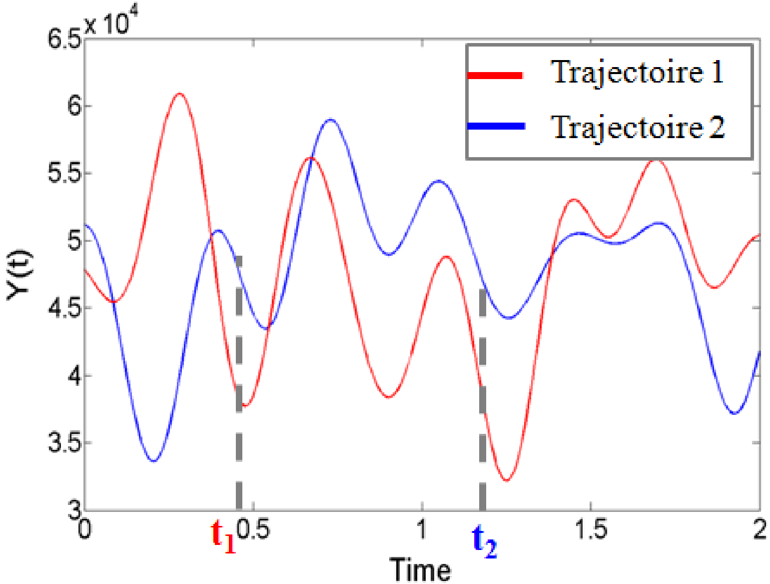


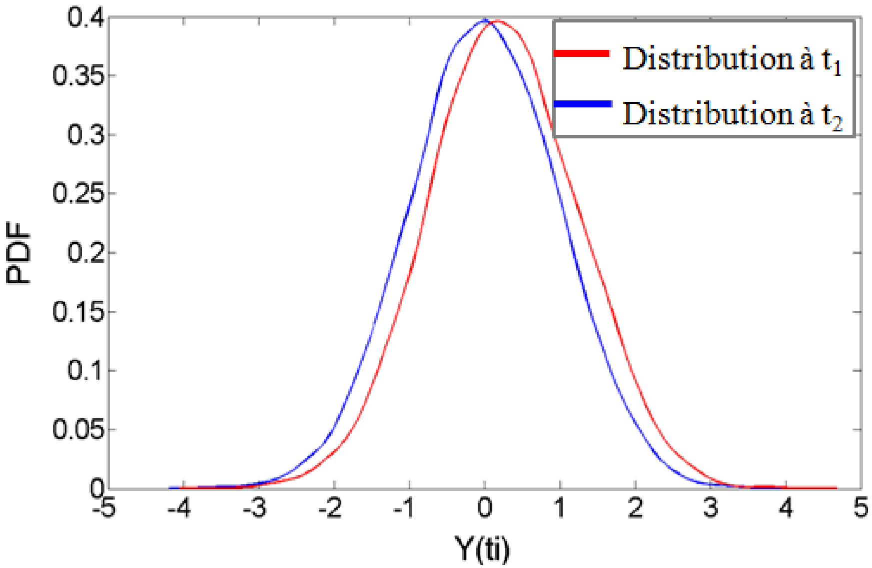


At two instants t_1 and t_2, the corresponding random variables $Y(t_1)$ and $Y(t_2)$ are correlated. The exponential square autocorrelation function is commonly used:

$$\rho_Y(t_1, t_2) = e^{-\left(\frac{t_2 - t_1}{\ell}\right)^2} \quad , \ell \text{ = autocorrelation length}$$

Stochastic process *Y*(*t*)



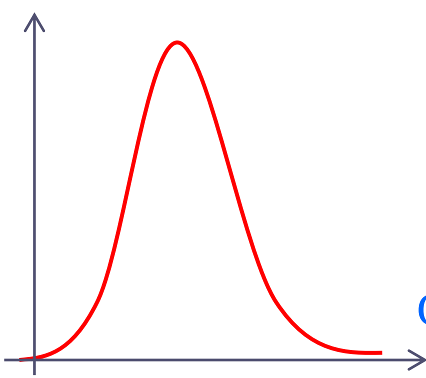


Stationary
or
Non-Stationary

Gaussian
or
Non-Gaussian

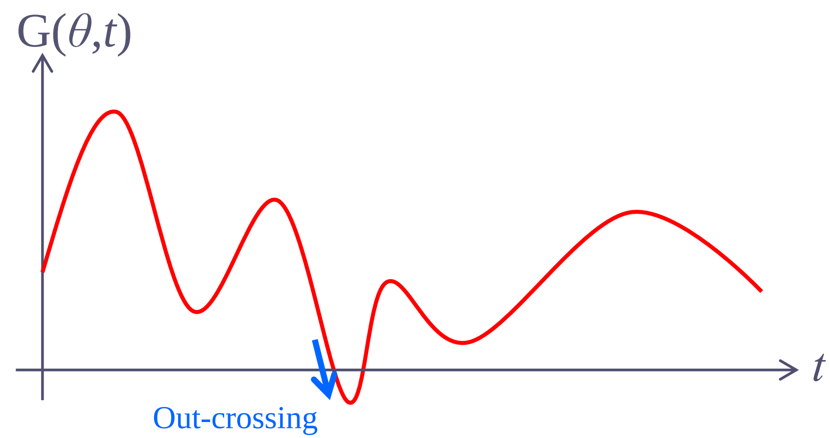
Extreme performance approach

Sample	Time	G_{min}
X_1	t_1^*	$G(t_1^*)$
X_2	t_2^*	$G(t_2^*)$
.	.	.
.	.	.
X_N	t_N^*	$G(t_N^*)$



$$P_{f,c}(t_i,t_f) \approx \text{Prob}\,(\min_{t_i \le \tau \le t_f} \{G(\tau)\} < 0)$$

Outcrossing approach



$$\upsilon^+(t) = \lim_{\Delta t \to 0^+} \frac{\text{Prob}(\{G(t) > 0\} \cap \{G(t+\Delta t) < 0\})}{\Delta t}$$

$$P_{f,c}(t_i,t_f) \le P_{f,i}(t_0) + E[N^+(t_i,t_f)]$$

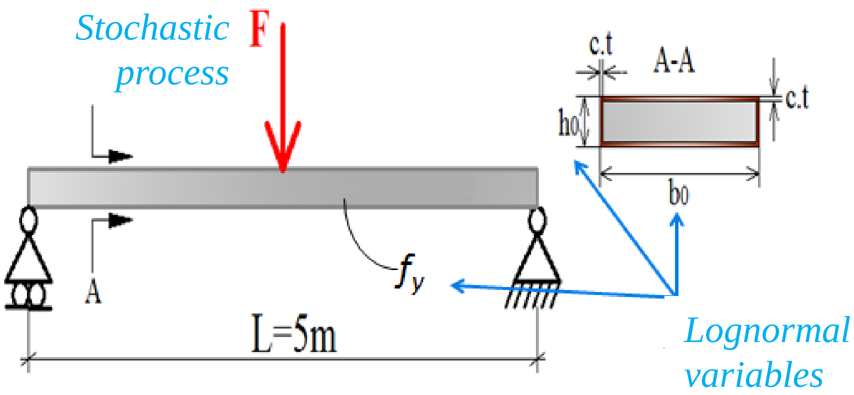
"PHI2 method"

Andrieu-Renaud C., Sudret, B. and Lemaire, L. (2004). "The PHI2 method: a way to compute time-variant reliability." Reliability Engineering and system Safety. 84: 75-86.

Main challenges

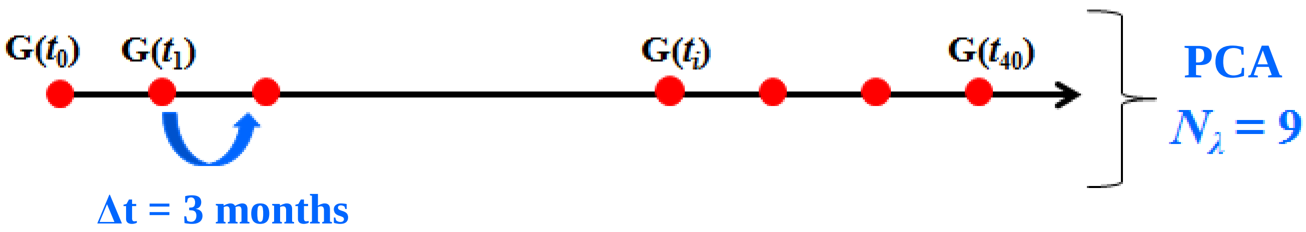
- requires a very high number of evaluations of the deterministic mechanical model
- existent time-variant reliability methods have some limitations (nonlinear limit state functions, computationally demanding global optimization)
- high dimensionality of time-dependent problems (discretization of stochastic processes)

Case study 1



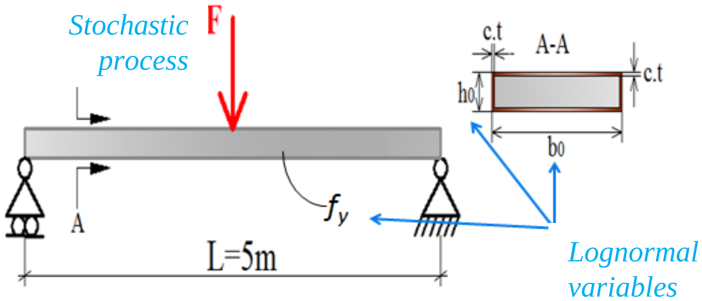
$$G(t) = \mathcal{M}_u(t) - \mathcal{M}_{max}(t) = \frac{(b_0 - 2ct)\,(h_0 - 2ct)^2}{4}\, f_y - \left[\frac{F(t)L}{4} + \rho_{st} \frac{b_0\, h_0\, L^2}{8} \right]$$

$$\rho_F(t_1,t_2) = exp\left(-\left(\frac{t_2 - t_1}{1\,an}\right)^2\right)$$

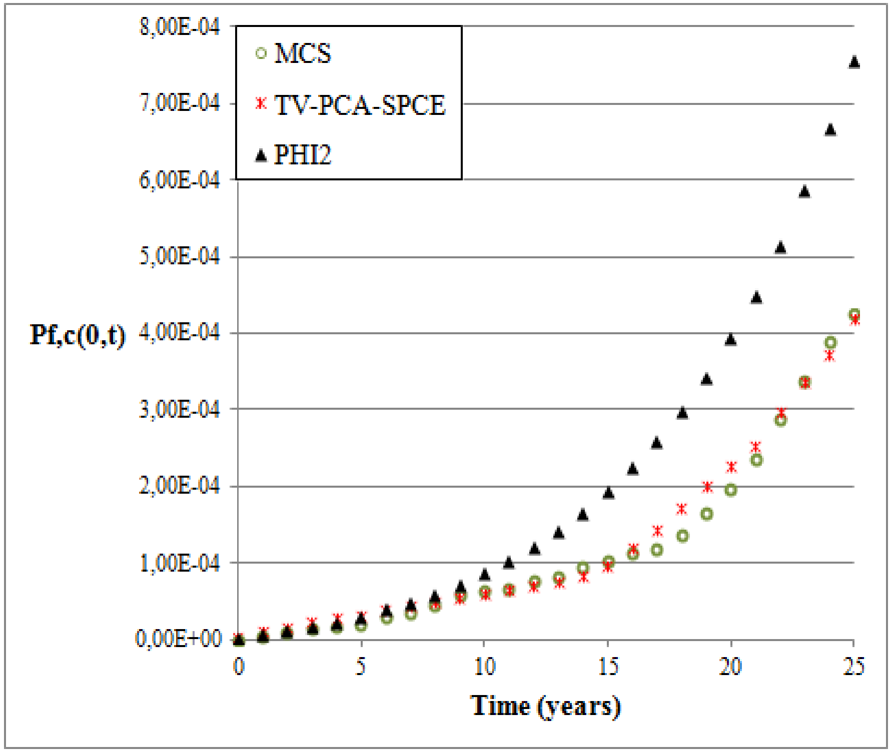


Illustrative Examples

Case study 1



	Number of function evaluations
MCS	1,000,000 x **100**
PHI2	18,720
TV-PCA-SPCE	200 x **100**



- Evolution in time of the cumulative probability of failure -

The proposed method (TV-PCA-SPCE):

- ✓ is very accurate,
- ✓ is more efficient than PHI2 (better accuracy with comparable computational costs.